Fredholm method for scars

1996 J. Phys. A: Math. Gen. 29 919

(http://iopscience.iop.org/0305-4470/29/4/019)

J. Phys. A: Math. Gen. **29** (1996) 919–937. Printed in the UK

Fredholm method for scars

S Fishman†, B Georgeot‡ and R E Prange
Department of Physics, University of Maryland, College Park, MD 20742, USA

Received 4 August 1995

Abstract. A new quasiclassical formula for scars is obtained by using the Fredholm method. We show that it can be expanded into a formula obtained earlier by Agam and Fishman. The derivation is simple and direct. It is also more rigorous and more general than that of Agam and Fishman. It also clarifies the remarkable process of resurgence, relating the high-order terms based on long orbits to the Weyl term whose origin is the zero length orbits.

1. Introduction

The semiclassical or quasiclassical approximation (QCA) has been an important and constantly used tool since the discovery of quantum mechanics. Although Einstein [1] presciently realized that there was a fundamental difficulty for classically non-integrable systems, it is only relatively recently [2–4] that it has been widely appreciated that the classically chaotic systems require a much more profound treatment than that based on the example of an integrable system.

The central difficulty is the *divergence* of the series that the semiclassical approximation produces. Much important work has been devoted in the past few years to overcome this difficulty, using, e.g., zeta functions [5, 6], cycle expansions [7] or the use of functional relations and analyticity on the series (in analogy to the Riemann–Siegel method for the Riemann zeta function) [8]. This variety of methods led to very good results for a particular class of problem, the *spectrum* of two-dimensional *closed* systems (usually *billiards*) which display *hard chaos*. (For hard chaos, all classical orbits are hyperbolically unstable.)

Although it was natural to begin the attack on the divergence problem by taking the simplest case of the spectrum, there are many important quantities of interest which are not spectral properties. One of the most remarkable observations of this type is that of *scars* [9, 10]. Scars are enhancements, sometimes spectacular, of quantum wavefunctions around a periodic orbit of the underlying classical system. They are observed in brute force numerical calculations of wavefunctions as well as experiment [11]. In order to study this effect, a semiclassical formula for individual wavefunctions using the Berry–Keating resummation scheme [8] has been developed in [12]. The formula predicts with quite good accuracy the existence and position of scars from purely classical input [13].

Recently, in order to generalize further the resummation methods, it has been shown in [14, 15], that use of the *Fredholm method* applied to the transfer operator of Bogomolny [16] gives a general resummation scheme for all quantities given by semiclassical series, such as scattering amplitudes and Green functions.

† Permanent address: Department of Physics, Technion, Haifa, 32000 Israel.
‡ Present address: Niels Bohr Institutet, Blegdamsvej 17, DK-2100, Copenhagen.

0305-4470/96/040919+19$19.50

In this paper the Fredholm method is applied to the Wigner function, giving a rather compact, general and rigorous result. As we shall see, for the case of hard chaos, the result can be shown to be the same as in [12]. (This paper of Agam and Fishman will be referred to as AF.) The justification for this new derivation is its simplicity and transparency, as well as its more rigorous setting. In particular, one sees a quite striking manifestation of the phenomenon of *resurgence*, in which the contribution of the long orbits is qualitatively and quantitatively related to the contribution of the zero length orbits.

All the equations in this paper are written for the case of two space dimensions. More general formulae can be written at a small cost of notational simplicity. Whether the higher dimensional version is useful and accurate remains for future work to assess.

2. Fredholm method for the Green function

Fredholm theory gives the solution of a certain class of integral equations, or alternatively, the solution of a class of operator equations [17]. A *Fredholm integral equation* of the second kind is

$$x(q) = x_0(q) + \lambda \int \mathrm{d}q'\, K(q, q')x(q')\, \mathrm{d}q' \tag{1}$$

which is equivalent to an operator equation, $x = x_0 + \lambda \boldsymbol{K} x$. In equation (1), the unknown function $x(q)$, and the known functions x_0 and K are defined on some finite domain (e.g. an interval of reals). If the functions, $x_0(q)$, $K(q, q')$ are sufficiently nice (e.g. continuous, or square integrable), the *Fredholm alternative* holds. That is, *either* there is a unique solution of equation (1), with the same nice properties, *or* the homogeneous version of equation (1) ($x_0 \equiv 0$) has a solution. There is a discrete set of complex eigenparameters $\lambda = \lambda_n$ for which the solution $x(q)$ is not unique.

In operator language, x and x_0 are elements of a Hilbert or Banach space, and $\boldsymbol{K}$ is a (compact) operator on that space. In this terminology, it is said that the inverse operator $[\mathbf{1} - \lambda \boldsymbol{K}]^{-1}$ exists except for a discrete set of λ's.

The kernel $K(q, q')$ can also be regarded as a continuous or infinite-dimensional matrix. Most of the schemes for numerical solution of equation (1) exploit this by making a discretization or truncation, which reduces the problem to inversion of a finite matrix. In this case $[\mathbf{1} - \lambda \boldsymbol{K}]^{-1}$ can be expressed as a ratio

$$\frac{1}{\mathbf{1} - \lambda \boldsymbol{K}} = \frac{\boldsymbol{N}(\lambda)}{D(\lambda)}\,. \tag{2}$$

If $\boldsymbol{K}$ is approximated as an $N \times N$ matrix, the determinant $D(\lambda) = \det(\mathbf{1} - \lambda \boldsymbol{K})$ is an Nth-order polynomial in λ whose N zeros give an approximation to the λ_n. The numerator is a polynomial of order $N - 1$.

The main result of Fredholm theory is that, when $\boldsymbol{K}$ is a compact operator, the expression (2) continues to hold, but with $D(\lambda)$ an *entire* function of λ. i.e. it is a series *absolutely convergent* for all $|\lambda| < \infty$ rather than a polynomial. Similarly, the numerator is an *operator valued* entire function of λ. We give the explicit expressions for the Fredholm determinant D and the numerator operator $\boldsymbol{N}$ in terms of the kernel $\boldsymbol{K}$ below.

We wish to apply this method to the energy Green function, $G(\boldsymbol{r}, \boldsymbol{r}', E)$, which can be considered as the inverse operator $[E - \boldsymbol{H} + \mathrm{i}\eta]^{-1}$, where $\boldsymbol{H}$ is the Hamiltonian, i.e.

$$G(\boldsymbol{r}, \boldsymbol{r}', E) = \left\langle \boldsymbol{r} \left| \frac{\mathbf{1}}{E - \boldsymbol{H} + \mathrm{i}\eta} \right| \boldsymbol{r}' \right\rangle. \tag{3}$$

Once we obtain G it is straightforward to find the Wigner distribution.

In QCA the (outgoing wave) Green function [18] is

$$G(\boldsymbol{r},\boldsymbol{r}',E)=\frac{1}{(2\pi\mathrm{i}\hbar)^{1/2}}\frac{1}{\mathrm{i}\hbar}\sum_{\text{class traj}}\frac{1}{\sqrt{vv'}}\sqrt{\left|\frac{\partial^2 S}{\partial y\partial y'}\right|}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\boldsymbol{r},\boldsymbol{r}',E)-\mathrm{i}\nu\frac{\pi}{2}}. \tag{4}$$

The sum is over all classical trajectories going from $\boldsymbol{r}'$ to $\boldsymbol{r}$ at energy E. The action S is the integral $\int \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{r}$ along a trajectory; $\partial^2 S/\partial y\partial y'$ is the cross derivative of the action with respect to the coordinates perpendicular to the trajectory at $\boldsymbol{r}$ and $\boldsymbol{r}'$ and v, v' are the speeds at $\boldsymbol{r}, \boldsymbol{r}'$. The index ν counts the conjugate points along the trajectory. This sum does not converge. Indeed, it could not be a good approximation to equation (3) if it did, since the exact Green function must be infinite at an eigenenergy. We may consider it, however, for energies E with a sufficiently large positive imaginary part and eventually analytically continue it to real energy.

In two or more dimensions (the real part of) $G(\boldsymbol{r},\boldsymbol{r},E)$ is infinite, even if E is not at an eigenenergy. The Gutzwiller trace formula [2], is obtained by considering $\mathrm{Tr}\,\mathrm{Im}\,G=\mathrm{Im}\int G(\boldsymbol{r},\boldsymbol{r},E)\,\mathrm{d}\boldsymbol{r}$.

An intuitive trick to get an integral equation for equation (4) is to use a *quantum surface of section*, first introduced by Bogomolny [16]. This may be rigorously justified in most cases by generalizing a result of Prozen [19], which in turn is a generalization of a method of Doron and Smilansky [6]. The *Poincaré surface of section* (SS) is a very common tool in classical mechanics: we specialize it slightly to be a phase space surface whose configuration space has one dimension less than the original configuration space, and whose phase space component is the momentum conjugate to the configuration space. It is most conveniently chosen such that a typical classical trajectory will cross it after a finite time. Originally [16], the SS was defined only classically and semiclassically, by use of an operator, called the *transfer operator* $\boldsymbol{T}$ which acts on the space of functions defined on the configuration part of the SS. It can be thought of as a kernel, $T(q,q')$, relating two points on the SS by means of the classical trajectories between these points. A very important property of this operator is that it is approximately finite (of finite rank), i.e. very close to a finite matrix, and approximately unitary for closed systems [16].

In [14, 15] it was shown that for many quantities, such as Green functions or scattering amplitudes, the semiclassical series is essentially an approximation of the inverse $[\mathbf{1}-\boldsymbol{T}]^{-1}$. Therefore, it is natural to apply the *Fredholm method* since $\boldsymbol{T}$ is typically a *compact* operator, that is, $T(q,q')$ is continuous or square integrable. Here the aim is to apply it to the Wigner transform of the resolvent to get a resummed formula for the wavefunctions. The first step is to write the Green function, or resolvent, in this formalism.

The semiclassical transfer operator is given by Bogomolny [16] as

$$T(q,q')=\frac{1}{(2\pi\mathrm{i}\hbar)^{1/2}}\left|\frac{\partial^2 S}{\partial q\partial q'}\right|^{1/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(q,q',E)-\mathrm{i}\nu\frac{\pi}{2}}. \tag{5}$$

Here q and q' label distance along the SS. The SS is one dimensional, i.e. one less than the two-dimensional space of the original problem. $S(q,q')$ is the action integral of the orbit of energy E leaving $q\prime$ and returning for the first time to the surface of section at q. We have shown just one such contribution, since it is usually possible and convenient to choose the SS so that there is at most one orbit contributing to $T(q,q')$. Otherwise there will be a sum over those classical trajectories of 'first return' going from q' to q. (It is customary to insist that the first return be in the 'positive' direction, and that the orbit leave q' in the 'positive' direction. We shall also count as SS crossings only those in the positive direction). It is

easy to check [16] that

$$T^2(q,q'') = \int T(q,q')T(q',q'')\,\mathrm{d}q' = \frac{1}{(2\pi \mathrm{i}\hbar)^{1/2}} \sum_{\text{class traj}} \left|\frac{\partial^2 S}{\partial q\,\partial q''}\right|^{1/2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(q,q'',E)-\mathrm{i}\nu\frac{\pi}{2}} \tag{6}$$

where the sum is over classical trajectories of *second return* going from q' to q, provided the integral is performed by the stationary phase approximation, SΦ. There will typically be more than one point of stationary phase, thus more than one classical trajectory for trajectories of second return, if the classical dynamics is chaotic. (The number of such trajectories returning n times then grows exponentially with n.)

We define closely related kernels by

$$V_+(q',\boldsymbol{r}') = \frac{1}{\sqrt{\mathrm{i}\hbar}}\frac{1}{\sqrt{2\pi\mathrm{i}\hbar}}\frac{1}{\sqrt{v'}}\left|\frac{\partial^2 S}{\partial q'\partial y'}\right|^{1/2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(q',\boldsymbol{r}',E)-\mathrm{i}\nu\frac{\pi}{2}} \tag{7}$$

where we have assumed there is just one classical trajectory going from a point $\boldsymbol{r}'$ anywhere in the system to a point q' on the SS . The trajectory must also arrive at the SS for the first time in the positive direction. The derivatives of the action are taken with respect to q' and the coordinate orthogonal to the trajectory at $\boldsymbol{r}'$. Similarly one can define

$$V_-(\boldsymbol{r},q) = \frac{1}{\sqrt{\mathrm{i}\hbar}}\frac{1}{\sqrt{2\pi\mathrm{i}\hbar}}\frac{1}{\sqrt{v}}\left|\frac{\partial^2 S}{\partial y\,\partial q}\right|^{1/2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\boldsymbol{r},q,E)-\mathrm{i}\nu\frac{\pi}{2}}, \tag{8}$$

where the classical trajectory goes from a point of the SS q to a point $\boldsymbol{r}$ anywhere in the system, this time leaving the SS in the positive direction, and never crossing it again. These operators are illustrated in figure 1.

V_+ and V_- are operators which relate a point of the system to or from the SS. It is easy to see that $\int V_-(\boldsymbol{r},q)T^n(q,q')V_+(q',\boldsymbol{r}')\,\mathrm{d}q\,\mathrm{d}q'$, calculated in SΦ, corresponds to a sum over classical trajectories crossing the SS exactly $n+1$ times while going from $\boldsymbol{r}'$ to $\boldsymbol{r}$. Moreover, the prefactors are exactly the same as in equation (4). Therefore one can

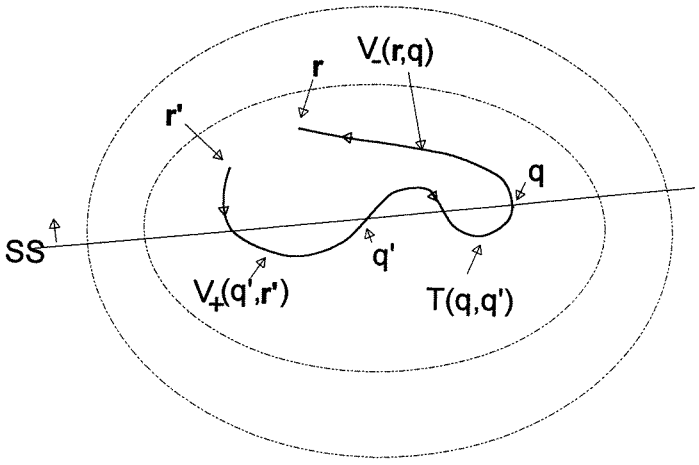

Figure 1. An orbit in a potential (schematically indicated by contour lines). The orbit starts at point $\boldsymbol{r}'$ and passes through the surface of section at q', again at q, and ends at $\boldsymbol{r}$, corresponding to $V_+(q',\boldsymbol{r}')T(q,q')V_-(\boldsymbol{r},q)$.

enumerate all the trajectories in equation (4) with respect to the number of times they cross the SS. One obtains

$$G(\boldsymbol{r},\boldsymbol{r}',E)=G_0(\boldsymbol{r},\boldsymbol{r}',E)+\sum_{n=0}^{\infty}\int V_-(\boldsymbol{r},q)T^n(q,q')V_+(q',\boldsymbol{r}')\,\mathrm{d}q\,\mathrm{d}q' \tag{9}$$

where G_0 corresponds to the direct trajectories between $\boldsymbol{r}$ and $\boldsymbol{r}'$, which do not cross the SS. We say that an orbit coming from the term T^n has SS *length* $n+1$. Direct orbits have SS length zero, and orbits from the term $\boldsymbol{V}_-\boldsymbol{V}_+$ have SS length one. We denote the SS length of an orbit p by the integer L_p. The expression (9) contains the sum of a geometric series $\sum \boldsymbol{T}^n$ which is formally equal to $[\mathbf{1}-\boldsymbol{T}]^{-1}$.

The *Fredholm determinant* $D(\lambda)$ for $[\mathbf{1}-\lambda\boldsymbol{T}]$ is given by the absolutely convergent series [17]

$$D(\lambda)=\sum_{n=0}^{\infty}\lambda^n D_n \tag{10}$$

where

$$D_n=\frac{(-1)^n}{n!}\int\!\!\int_{\mathrm{SS}}\mathrm{d}q_1\ldots\mathrm{d}q_n\det\begin{vmatrix}T(q_1q_1)&\cdots&T(q_1q_n)\\ \vdots&\ddots&\vdots\\ T(q_nq_1)&\cdots&T(q_nq_n)\end{vmatrix}. \tag{11}$$

Similarly, one expresses

$$\boldsymbol{N}(\lambda)=\sum_{n=0}^{\infty}\lambda^n\boldsymbol{N}_n \tag{12}$$

where $\boldsymbol{N}_n$ is an operator corresponding to the kernel

$$N_{n+1}(q,q')=D_{n+1}\delta(q-q')+\frac{(-1)^n}{n!}\times\int\!\!\int_{\mathrm{SS}}\mathrm{d}q_1\ldots\mathrm{d}q_n\det\begin{vmatrix}T(q,q')&T(q,q_1)&\cdots&T(q,q_n)\\ T(q_1,q')&T(q_1,q_1)&\cdots&T(q_1q_n)\\ \vdots&\vdots&\ddots&\vdots\\ T(q_n,q')&T(q_n,q_1)&\cdots&T(q_nq_n)\end{vmatrix}. \tag{13}$$

As $D=\det(\mathbf{1}-\lambda\boldsymbol{T})$, the theory shows that the well known equality between the logarithm of the determinant of a matrix and the trace of its logarithm generalizes to this case, i.e.

$$D_n=-\frac{1}{n}\sum_{r=1}^{n}\sigma_r D_{n-r} \tag{14}$$

where

$$\sigma_r=\mathrm{Tr}\,\boldsymbol{T}^r=\int\!\!\int_{\mathrm{SS}}\mathrm{d}q_1\ldots\mathrm{d}q_r\,T(q_1,q_2)\ldots T(q_r,q_1)\,. \tag{15}$$

Also using the fact that $\boldsymbol{N}(\lambda)=D(\lambda)/(\mathbf{1}-\lambda\boldsymbol{T})$ one has

$$\boldsymbol{N}_n=\sum_{r=0}^{n}D_r\boldsymbol{T}^{n-r}\,. \tag{16}$$

If $\boldsymbol{T}$ is of finite rank, that is, it can be expressed as a bilinear sum of N functions, it is equivalent to a finite matrix of size N. Then the series will be truncated after the term of index N for D, and after the term of index $N-1$ for $\boldsymbol{N}$.

Using the Fredholm result, we have

$$G(\boldsymbol{r}, \boldsymbol{r}', E) = G_0(\boldsymbol{r}, \boldsymbol{r}', E) + \frac{1}{D(E)} \int V_-(\boldsymbol{r}, q) N(q, q'; \lambda = 1) V_+(q', \boldsymbol{r}') \, \mathrm{d}q \, \mathrm{d}q' \tag{17}$$

where $D(E)$ is the Fredholm determinant at $\lambda = 1$ regarded as a function of E.

One simple feature of (17) should be noticed. Since by (16), the operator part of $\boldsymbol{N}$ is just a power of $\boldsymbol{T}$, the $\iint \mathrm{d}q \, \mathrm{d}q'$ integrals in (17) can be done by SΦ with the result that *exactly the same expressions* occur in (4), but multiplied by factors D_s/D. Define

$$G_r\left(\boldsymbol{r}, \boldsymbol{r}', E\right) = \frac{1}{(2\pi \mathrm{i}\bar{h})^{1/2}} \frac{1}{\mathrm{i}\bar{h}} \sum_{p, L_p = r} \frac{1}{\sqrt{v v'}} \sqrt{\frac{\partial^2 S_p}{\partial y \partial y'}} \, \mathrm{e}^{\frac{\mathrm{i}}{\bar{h}} S_p(\boldsymbol{r}, \boldsymbol{r}', E) - \mathrm{i}\nu \frac{\pi}{2}} . \tag{18}$$

Then

$$G(\boldsymbol{r}, \boldsymbol{r}', E) = G_0(\boldsymbol{r}, \boldsymbol{r}', E) + \frac{1}{D(E)} \sum_{n=0}^{\infty} \sum_{s=0}^{n} D_{n-s}(E) G_{s+1}(\boldsymbol{r}, \boldsymbol{r}', E) . \tag{19}$$

If the series $\sum_{s=0}^{\infty} G_{s+1}$ converges, then it is legal to change the order of the n and s sums in (19), leading to $G = G_0 + \sum_{s=0}^{\infty} G_{s+1} D^{-1} \sum_{n=s}^{\infty} D_{n-s} = \sum_{s=0}^{\infty} G_s$. Therefore, we see that the Fredholm method is just a 'trivial' regrouping of terms in the original series. This regrouping, however, guarantees convergence, and converts a divergent series into a convergent one!

3. Resurgence

The operator $\boldsymbol{T}$ is typically of finite rank in QCA. In fact, consider the expansion

$$T(q, q') = \sum_{r,s=-\infty}^{\infty} \varphi_r(q) T_{rs} \bar{\varphi}_s(q') \tag{20}$$

where the φ_r's are a complete orthonormal set on the (configuration part of) the SS. Suppose, for example, that $\varphi_r(q) = \sqrt{1/L} \exp(-2\pi \mathrm{i} r q / L)$ where L is the length of the SS. Then,

$$T_{rs} = \iint \mathrm{d}q \, \mathrm{d}q' \frac{1}{L} \mathrm{e}^{-2\pi \mathrm{i} \frac{rq}{L}} \frac{1}{(2\pi \mathrm{i}\bar{h})^{1/2}} \sum_{\text{class traj}} \left| \frac{\partial^2 S}{\partial q \partial q'} \right|^{1/2} \mathrm{e}^{\frac{\mathrm{i}}{\bar{h}} S(q, q', E) - \mathrm{i}\nu \frac{\pi}{2}} \mathrm{e}^{2\pi \mathrm{i} \frac{sq'}{L}} . \tag{21}$$

The leading order QCA approximation to T_{rs} requires the SΦ conditions $2\pi r/L = p/\bar{h} \equiv (\partial S(q, q')/\partial q)/\bar{h}$ and $2\pi s/L = p'/\bar{h} \equiv -(\partial S(q, q')/\partial q')/\bar{h}$. Since p, p' are the classical momenta conjugate to q, q', and since energy is conserved, thus bounding $|p|$ and $|p'|$, it is seen that a leading order contribution will only occur for $|r| < L\,|p|\,/h$. If the maximum momentum corresponds to a minimum wavelength $\lambda_{\min}$ it is found that the QCA dimension of $\boldsymbol{T}$ is $N \times N$ where $N = 2L/\lambda_{\min}$.

Furthermore, $\boldsymbol{T}$ is unitary in the semiclassical approximation (i.e. the corrections to unitarity are order $\bar{h}^2$) [16]. It is instructive to be more precise on this point. The preceding argument shows that $\boldsymbol{T}$ has rank N, which means that there is a representation (e.g. the momentum representation), where all the non-zero elements T_{rs} lie in an $(N \times N)$-dimensional space. (The rank of the matrix N may actually be less than $2L/\lambda_{\min}$, which is an upper bound.) Let $\boldsymbol{P}$ be the projection operator onto this space. Then $\boldsymbol{T}(\boldsymbol{1} - \boldsymbol{P})$ and $(\boldsymbol{1} - \boldsymbol{P})\boldsymbol{T}$ vanish in QCA. The 'unitarity' property is more precisely $\boldsymbol{T}^\dagger \boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}^\dagger = \boldsymbol{T}\boldsymbol{P}\boldsymbol{T}^\dagger = \boldsymbol{P}$. Remark as well that $\boldsymbol{P}\boldsymbol{V}_+ = \boldsymbol{V}_+$ and $\boldsymbol{V}_- = \boldsymbol{V}_-\boldsymbol{P}$. We refer to this property of unitarity restricted to the subspace defined by $\boldsymbol{P}$ simply as unitarity.

In other representations for $\boldsymbol{T}$, e.g. the discretized coordinate representation, this property will not be so obvious. In that case, in numerical practice, $\boldsymbol{T}$ will have N eigenvalues approximately on the unit circle, with the rest quasiclassically small. There also may be an eigenvalue of intermediate magnitude, since the matrix must change size as a function of energy. However, one may replace the approximate $\boldsymbol{T}$ by one exactly satisfying the unitarity property, making quasiclassical errors only. We suppose that this has been done in the sequel. We mean by $\boldsymbol{T}^{-1}$ in this case, the operator which is inverse to $\boldsymbol{T}$ on the subspace defined by $\boldsymbol{P}$, and which vanishes elsewhere.

The unitarity property implies that the second half of the terms in the finite series for $D(\lambda)$ and $\boldsymbol{N}(\lambda)$, which are usually the most difficult to compute, can be expressed as a function of the first half. This greatly simplifies the computations, and is similar to the property of *resurgence* of an infinite series, which in mathematics relates the high-order terms of an infinite series to the low-order ones. This property was first exploited in the case of the quasiclassical series in [8]. Using unitarity, relation $\overline{D(\lambda)} = \det(\mathbf{1} - \bar{\lambda}\boldsymbol{T}^{-1}) = D(\bar{\lambda}^{-1})\bar{\lambda}^{N}\mathrm{e}^{-\mathrm{i}\Phi}$ has been established [16]. Here $\mathrm{e}^{\mathrm{i}\Phi} = \det(-\boldsymbol{T} + \mathbf{1} - \boldsymbol{P})$. (This determinant is $(-1)^N$ times the product of the eigenvalues of $\boldsymbol{T}$ which are on the unit circle. The term $\mathbf{1} - \boldsymbol{P}$ is the projector on the space where $\boldsymbol{T}$ vanishes.) This implies $D_n = \bar{D}_{N-n}\mathrm{e}^{\mathrm{i}\Phi}$. The phase is known to be $\Phi = 2\pi\mathcal{N}$ where $\mathcal{N}(E)$ is the smoothed (Weyl) spectral staircase function, i.e. the number of energy levels less than E.

The relation for $\boldsymbol{N}(\lambda)$ was shown in [14]; one uses the fact that $\boldsymbol{N}(\lambda) = D(\lambda)/(\mathbf{1} - \lambda\boldsymbol{T})$ so $\boldsymbol{P}\boldsymbol{N}(\lambda)^{\dagger} = -\bar{D}[\bar{\lambda}^{-1}\boldsymbol{T}/(1 - \bar{\lambda}^{-1}\boldsymbol{T})] = -\bar{\lambda}^{N-1}\boldsymbol{T}\mathrm{e}^{-\mathrm{i}\Phi}\boldsymbol{N}(\bar{\lambda}^{-1})$. Expanding in powers of $\bar{\lambda}$ and equating coefficients gives

$$\boldsymbol{P}\boldsymbol{N}_n = -\boldsymbol{T}^{\dagger}\mathrm{e}^{\mathrm{i}\Phi}\boldsymbol{N}_{N-n-1}^{\dagger}\,. \tag{22}$$

In particular, we can express $\boldsymbol{P}\boldsymbol{N}_{N-1} = -\boldsymbol{T}^{\dagger}\mathrm{e}^{\mathrm{i}\Phi}$.

The denominator is

$$D = \sum_{n=0}^{N} D_n\,. \tag{23}$$

We take N odd, $N = 2N_1 + 1$, to be specific. The formulae for even N are similar. Applying the relation between high-order and low-order terms gives

$$D = \sum_{n=0}^{N_1} D_n + \sum_{n=N_1+1}^{2N_1+1} \mathrm{e}^{\mathrm{i}\Phi} D_{N-n}^{*} \tag{24}$$

i.e

$$D = \mathrm{e}^{\mathrm{i}\Phi/2} \sum_{n=0}^{N_1} (\mathrm{e}^{-\mathrm{i}\Phi/2} D_n + \mathrm{e}^{\mathrm{i}\Phi/2} D_n^{*})\,. \tag{25}$$

If

$$\tilde{\Delta} \equiv \mathrm{e}^{-\mathrm{i}\Phi/2} \sum_{n=0}^{N_1} D_n \tag{26}$$

then

$$\mathrm{e}^{-\mathrm{i}\Phi/2} D \equiv \Delta = \tilde{\Delta} + \tilde{\Delta}^{*} = 2\,\mathrm{Re}\,\tilde{\Delta} \tag{27}$$

i.e. the *spectral determinant* Δ [8] is manifestly real.

One can do exactly the same for the numerator, using the relation (22). Then (restricting ourself to the space where $\boldsymbol{T}$ is unitary)

$$\boldsymbol{N} = \sum_{n=0}^{2N_1} \boldsymbol{N}_n = \sum_{n=0}^{N_1-1} \boldsymbol{N}_n - \sum_{n=N_1}^{2N_1} \boldsymbol{T}^\dagger \mathrm{e}^{\mathrm{i}\Phi} \boldsymbol{N}_{N-n-1}^\dagger \,. \tag{28}$$

Here the sum terminates at $N-1$ and not N, in contrast with the determinant.

Inserting equation (28) into expression (17), we see that we must find $\boldsymbol{V}_-\boldsymbol{T}^\dagger$:

$$\int V_-(\boldsymbol{r}, q) T^\dagger(q, q')\, \mathrm{d}q = \frac{1}{(2\pi \mathrm{i}\hbar)} \frac{1}{\sqrt{\mathrm{i}\hbar}} \times \int \frac{1}{\sqrt{v}} \left| \frac{\partial^2 S}{\partial y \partial q} \right|^{1/2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\boldsymbol{r}, q, E)} \left| \frac{\partial^2 S}{\partial q \partial q'} \right|^{1/2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} S(q', q, E)}\, \mathrm{d}q \tag{29}$$

where we have assumed as before that just one such short classical trajectory exists. The SΦ approximation selects the path such that

$$\frac{\partial S(\boldsymbol{r}, q)}{\partial q} - \frac{\partial S(q', q)}{\partial q} = p(q', q) - p(\boldsymbol{r}, q) = 0 \,. \tag{30}$$

Thus, the classical path starting at q and continuing to q' begins with the same momentum as the path starting at q and going to $\boldsymbol{r}$ (note that here we discuss the meaning of $\boldsymbol{T}^\dagger$, not $\boldsymbol{T}$, so the geometry of figure 1 does not apply). Since the dynamics is deterministic, this singles out the path (for simplicity assumed unique), which begins at q then passes through $\boldsymbol{r}$ on its way to q'. The actions in the part between $\boldsymbol{r}$ and q of the paths exactly cancel, leaving only the (negative) action between q' and $\boldsymbol{r}$. One can check that

$$\frac{(\partial^2 S(\boldsymbol{r}, q)/\partial y \partial q)(\partial^2 S(q', q)/\partial q' \partial q)}{(\partial^2 S(\boldsymbol{r}, q)/\partial^2 q) - (\partial^2 S(q', q)/\partial^2 q)} = \frac{\partial^2 S(q', \boldsymbol{r})}{\partial y \partial q'} \,. \tag{31}$$

This leads to the interesting result

$$\mathrm{i} \int V_-(\boldsymbol{r}, q) T^\dagger(q, q')\, \mathrm{d}q = V_+^\dagger(\boldsymbol{r}, q') \,. \tag{32}$$

A similar SΦ computation shows that

$$\mathrm{i} \int T^\dagger(q', q) V_+(q, \boldsymbol{r})\, \mathrm{d}q = V_-^\dagger(q', \boldsymbol{r}) \,. \tag{33}$$

It is convenient to have *two* powers of $\boldsymbol{T}^\dagger$ rather than one in the resurgent terms, i.e. the second term on the right-hand side of (28). This is achieved by using

$$\boldsymbol{N}_n = D_n \mathbf{1} + \boldsymbol{T} \boldsymbol{N}_{n-1} \tag{34}$$

for $n \neq 0$ and $\boldsymbol{N}_0 = D_0 \mathbf{1}$.

This gives

$$\boldsymbol{N} = \sum_{n=0}^{2N_1} \boldsymbol{N}_n = \sum_{n=0}^{N1-1} \boldsymbol{N}_n - \sum_{n=1}^{N_1} \boldsymbol{T}^\dagger \mathrm{e}^{\mathrm{i}\Phi} (\boldsymbol{T}^\dagger \boldsymbol{N}_{n-1}^\dagger + \bar{D}_n) - \boldsymbol{T}^\dagger \mathrm{e}^{\mathrm{i}\Phi} \boldsymbol{N}_0^\dagger \tag{35}$$

$$= \sum_{n=0}^{N_1-1} (\boldsymbol{N}_n - (\boldsymbol{T}^\dagger)^2 \mathrm{e}^{\mathrm{i}\Phi} \boldsymbol{N}_n^\dagger) - \sum_{n=0}^{N_1} \boldsymbol{T}^\dagger \mathrm{e}^{\mathrm{i}\Phi} \bar{D}_n \,. \tag{36}$$

Using this in (17), we find (with $\lambda = 1$) that

$$G(\boldsymbol{r}, \boldsymbol{r}', E) = G_0(\boldsymbol{r}, \boldsymbol{r}', E) + g_L(\boldsymbol{r}, \boldsymbol{r}', E) + g_R(\boldsymbol{r}, \boldsymbol{r}', E) \,. \tag{37}$$

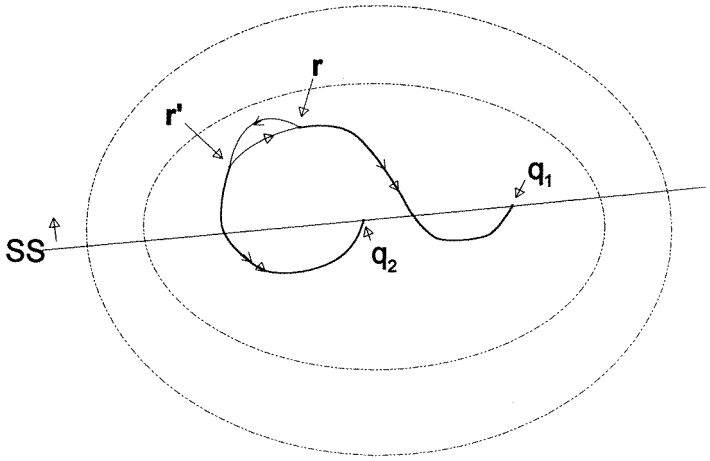

Figure 2. Orbits in the resurgent term g_R. The orbit from q_2 to q_1 is shown, labelled with the closed arrowheads. The part from $\boldsymbol{r}$ to q_1, labelled by the open arrowhead, is cancelled by the factor $V_+^\dagger(\boldsymbol{r}, q_1)$. Similarly, the orbit from $\boldsymbol{r}$ through $\boldsymbol{r}'$ to q_2 is shown. Assuming time reversal in not an invariance, the orbit from $\boldsymbol{r}$ to $\boldsymbol{r}'$ is not the time-reversed orbit from $\boldsymbol{r}'$ to $\boldsymbol{r}$.

This formula has three terms. The first corresponds to direct trajectories from $\boldsymbol{r}'$ to $\boldsymbol{r}$, the second to longer orbits going from $\boldsymbol{r}'$ to $\boldsymbol{r}$ while crossing the SS some positive number of times. The third, as we shall see, is a rather spectacular example of resurgence, since on the one hand, it comes from the longest orbits, and on the other hand, its contribution is almost the same as that of the direct orbits in G_0.

The third term in (37) is

$$g_R \equiv -\frac{\tilde{\Delta}^*}{\Delta} \int V_-(\boldsymbol{r}, q) T^\dagger(q, q') V_+(q', \boldsymbol{r}') \, \mathrm{d}q \, \mathrm{d}q' \tag{38}$$

which is by use of (32)

$$g_R = \mathrm{i} \frac{\tilde{\Delta}^*}{\Delta} \int V_+^\dagger(\boldsymbol{r}, q') V_+(q', \boldsymbol{r}') \, \mathrm{d}q' \,. \tag{39}$$

The integral is again performed in SΦ. There are two stationary phase points and we write their contributions as

$$g_R = -\frac{\tilde{\Delta}^*}{\Delta} [g_{R1} + g_{R2}] \,. \tag{40}$$

These SΦ points have the following geometric meaning, which is illustrated in figure 2. Consider the direct orbit from $\boldsymbol{r}'$ to $\boldsymbol{r}$ (which is assumed unique). Continue this orbit to the point q_1 where it first crosses the surface of section in the positive direction. The action of the factor $V_+(q_1, \boldsymbol{r}')$ is $S(q_1, \boldsymbol{r}')$ while the action of the factor $V_+^\dagger(\boldsymbol{r}, q_1)$ is $-S(q_1, \boldsymbol{r})$. This gives the total action factor of the SΦ point q_1 as $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\boldsymbol{r}, \boldsymbol{r}')}$. This of course is just the action factor of the direct orbit between $\boldsymbol{r}'$ and $\boldsymbol{r}$. The prefactors work out to give

$$g_{R1}(\boldsymbol{r}, \boldsymbol{r}', E) = G_0(\boldsymbol{r}, \boldsymbol{r}', E) \,. \tag{41}$$

The second contribution is from the point q_2 which is found as the first intersection with the SS of the continuation of the orbit which passes from $\boldsymbol{r}$ through $\boldsymbol{r}'$. Note that in the absence of time-reversal invariance, this orbit is *not* the time reversal of the orbit from $\boldsymbol{r}'$ to $\boldsymbol{r}$. The

SΦ point q_2 is found to lead to an action factor $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}S(\boldsymbol{r}',\boldsymbol{r})}$. Only if there is time-reversal invariance is $S(\boldsymbol{r},\boldsymbol{r}') = S(\boldsymbol{r}',\boldsymbol{r})$. The prefactor works out to give

$$g_{R2}(\boldsymbol{r},\boldsymbol{r}',E) = -G_0^*(\boldsymbol{r}',\boldsymbol{r},E)\,. \tag{42}$$

Thus g_R, the contribution of the longest orbits which do not completely cancel against each other (except for the important spectral determinant factor) can be expressed in terms of short, direct orbits, up to corrections which are quasiclassically small.

Similarly, we may write the contribution of intermediate length orbits. The second term in (37) is

$$g_L(\boldsymbol{r},\boldsymbol{r}',E) = \frac{1}{D}\sum_{n=0}^{p-1}\iint V_-(\boldsymbol{r},q)\left[(\boldsymbol{N}_n - \mathrm{e}^{\mathrm{i}\Phi}\boldsymbol{T}^{-2}\boldsymbol{N}_n^\dagger)(q,q')\right]V_+(q',\boldsymbol{r}')\,\mathrm{d}q\,\mathrm{d}q'\,. \tag{43}$$

Using equations (32) and (33) one finds that

$$V_-\left(\boldsymbol{T}^\dagger\right)^{k+2}V_+ = -\left[V_-\boldsymbol{T}^kV_+\right]^\dagger\,. \tag{44}$$

With the help of this relation one finds

$$g_L = \sum_{n=0}^{N_1-1}(g_n + g_n^\dagger) \tag{45}$$

where

$$g_n(\boldsymbol{r},\boldsymbol{r}',E) = \frac{1}{D}\iint V_-(\boldsymbol{r},q)\boldsymbol{N}_n(q,q')V_+(q',\boldsymbol{r}')\,\mathrm{d}q\,\mathrm{d}q'\,. \tag{46}$$

We have thus found that the longest orbits are cancelled to quasiclassical accuracy by combinations of shorter ones, while orbits of length between $N/2$ and N are expressed in terms of orbits of length between 0 and $N/2$. These cancellations are, in this perspective, a nice property of determinants. For a more physical insight into this cancellation, see [7, 15, 20].

4. The Wigner distribution

The Green function has now been put in a resummed form and we may take its Wigner transform. The Wigner transform of G is

$$W(\boldsymbol{x},E) = \int \mathrm{d}\boldsymbol{r}'\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}'}G(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}',\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}',E) \tag{47}$$

where $\boldsymbol{x} = (\boldsymbol{p},\boldsymbol{r})$. The QCA Green function (4) is used in this expression. Under these assumptions the Wigner transform of (4) that is performed in the semiclassical approximation yields

$$W(\boldsymbol{x},E) = \sum_{\text{class traj}} W^{(\text{m.r.})}(\boldsymbol{x},E) \tag{48}$$

where the sum is restricted to the classical trajectories that satisfy the midpoint rule $\boldsymbol{x} = \frac{1}{2}(\boldsymbol{x}_f + \boldsymbol{x}_i)$ where $\boldsymbol{x}_f$ and $\boldsymbol{x}_i$ are the final and initial points in phase space, respectively, [10]. The terms on the RHS are the Wigner transforms of the corresponding terms on the RHS of (4). Grouping terms according to the number of crossings of the SS we introduce the semiclassical Wigner transform of (18)

$$W_r'(\boldsymbol{x},E) = \sum_{\text{class traj}} W^{(\text{m.r.})}(\boldsymbol{x},E) \tag{49}$$

where the sum is restricted to orbits that cross the SS r times and satisfy the midpoint rule. The resulting expression for the Wigner function corresponding to (19) is

$$W(\boldsymbol{x}, E) = W_0(\boldsymbol{x}, E) + \frac{1}{D(E)} \sum_{n=0}^{\infty} U_n(\boldsymbol{x}, E) \tag{50}$$

where

$$U_n(\boldsymbol{x}, E) = \sum_{r=0}^{n} D_{n-r}(E) W'_{r+1}(\boldsymbol{x}, E) \tag{51}$$

where the prime was omitted for W_0.

In the semiclassical approximation the sum (19) terminates at some finite $n = N - 1$. Taking, as in section 3, $N = 2N_1 + 1$ odd, we can obtain an approximate expression for the Wigner function in terms of a finite sum over crossings of the SS. It is easy to see that if $W'_r(\boldsymbol{x}, E)$ is the Wigner transform (47) of $G_r(\boldsymbol{r}, \boldsymbol{r}', E)$, then the Wigner transform of $G^*_r(\boldsymbol{r}', \boldsymbol{r}, E)$ is $W'^*_r(\boldsymbol{x}, E)$, where * denotes complex conjugation. If the Wigner transform of G_0 is W_0 then the Wigner transforms of g_{R1} and g_{R2} are W_0 and W^*_0, respectively. Therefore the expression for the Wigner function resulting from (37), (40) and (45) is

$$W(\boldsymbol{x}, E) = W_0(\boldsymbol{x}, E) + \frac{1}{D} \sum_{n=0}^{N_1} \sum_{r=0}^{n} \left[D_{n-r} W'_{r+1}(\boldsymbol{x}, E) + \mathrm{e}^{\mathrm{i}\Phi} D^*_{n-r} W'^*_{r+1}(\boldsymbol{x}, E) \right]$$
$$- \frac{1}{D} \left[W_0(\boldsymbol{x}, E) - W^*_0(\boldsymbol{x}, E) \right] \mathrm{e}^{\mathrm{i}\Phi} \sum_{n=0}^{p} D^*_n \,. \tag{52}$$

First we note that the contribution from the short orbits is

$$W^R_0(\boldsymbol{x}, E) = W_0(\boldsymbol{x}, E) - \frac{1}{D} \left[W_0(\boldsymbol{x}, E) - W^*_0(\boldsymbol{x}, E) \right] \mathrm{e}^{\mathrm{i}\Phi} \sum_{n=0}^{p} D^*_n \tag{53}$$

where the last term results from the resurgent orbits. One finds easily that

$$W^R_0(\boldsymbol{x}, E) = \frac{1}{\Delta} \left[W_0(\boldsymbol{x}, E) \tilde{\Delta} + W^*_0(\boldsymbol{x}, E) \tilde{\Delta}^* \right] \tag{54}$$

where $\Delta = D\mathrm{e}^{-\mathrm{i}\Phi/2}$ (see equation (27)) is the spectral determinant, while $\tilde{\Delta}$ is defined by (26).

The resulting formula for the Wigner function is

$$W(\boldsymbol{x}, E) = W^R_0(\boldsymbol{x}, E) + \frac{1}{\Delta} \sum_{n=0}^{N_1} \left[\mathrm{e}^{-\mathrm{i}\Phi/2} U_n(\boldsymbol{x}, E) + \mathrm{e}^{\mathrm{i}\Phi/2} U^*_n(\boldsymbol{x}, E) \right] \tag{55}$$

where U_n is defined by (51). This expression is manifestly real. We remark that the resurgent terms for both very short and longer orbits are needed to achieve this reality.

The Wigner function W_α of an eigenstate α with energy E_α is found from the residues of (see e.g. AF, equation (3.3)):

$$W(\boldsymbol{x}; E) = h^2 \sum_{\alpha} \frac{W_\alpha(\boldsymbol{x})}{E + \mathrm{i}\eta - E_\alpha} \,. \tag{56}$$

The resulting Wigner function of the eigenstate α is

$$W_\alpha(\boldsymbol{x}) = \frac{1}{h^2 \Delta'(E_\alpha)} \Bigg\{ \left[W_0(\boldsymbol{x}, E) \tilde{\Delta} + W^*_0(\boldsymbol{x}, E) \tilde{\Delta}^* \right]$$
$$+ \sum_{n=0}^{N_1} \left[\mathrm{e}^{-\mathrm{i}\Phi/2} U_n(\boldsymbol{x}, E) + \mathrm{e}^{\mathrm{i}\Phi/2} U^*_n(\boldsymbol{x}, E) \right] \Bigg\}_{E=E_\alpha} \tag{57}$$

where $\Delta'(E_\alpha)$ is the derivative of the spectral determinant with respect to the energy at E_α where it vanishes.

We turn now to a more detailed calculation of the contribution of the short orbits, i.e. those that do not cross the SS. This is just the Wigner transform (47) of G_0 defined by (9),

$$W_0 = \int \mathrm{d}\boldsymbol{r}'\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}'} G_0(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{r}', E)\,. \tag{58}$$

It takes the explicit form [10],

$$W_0(\boldsymbol{x}; E) = -\mathrm{i}\pi\,\{A(\boldsymbol{x}, E + \mathrm{i}\eta) - \mathrm{i}B(\boldsymbol{x}, E + \mathrm{i}\eta)\} \tag{59}$$

where

$$A(\boldsymbol{x}, E) = \frac{2}{\overline{h}|\frac{\ddot{\boldsymbol{x}}\wedge\dot{\boldsymbol{x}}}{\overline{h}}|^{1/3}}\mathrm{Ai}\left(\frac{2[\mathcal{H}(\boldsymbol{x}) - E]}{(\overline{h}^2\ddot{\boldsymbol{x}}\wedge\dot{\boldsymbol{x}})^{1/3}}\right) \tag{60}$$

and

$$B(\boldsymbol{x}, E) = \frac{2}{\overline{h}|\frac{\ddot{\boldsymbol{x}}\wedge\dot{\boldsymbol{x}}}{\overline{h}}|^{1/3}}\mathrm{Gi}\left(\frac{2[\mathcal{H}(\boldsymbol{x}) - E]}{(\overline{h}^2\ddot{\boldsymbol{x}}\wedge\dot{\boldsymbol{x}})^{1/3}}\right) \tag{61}$$

where $\mathcal{H}$ is the Hamiltonian while the phase space velocity is $\dot{\boldsymbol{x}}$ and the acceleration is $\ddot{\boldsymbol{x}}$. The Airy function $\mathrm{Ai}(z)$ and the function $\mathrm{Gi}(z)$ are defined as the real and the imaginary parts of the integral

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(\frac{1}{3}t^3 + zt)} \tag{62}$$

respectively. $A(x, E)$ is approximately a delta function of energy conservation, $\delta(\mathcal{H}(\boldsymbol{x}) - E)$, but in addition it describes the pattern of Airy fringes as $\boldsymbol{x}$ moves off the energy surface. Using equation (54) one finds

$$W_0^R(\boldsymbol{x}, E) = \frac{\pi}{\Delta}\,[A(\boldsymbol{x}, E)\Delta_i - B(\boldsymbol{x}, E)\Delta] \tag{63}$$

where $\Delta_i = 2\ \mathrm{Im}\,\tilde{\Delta}$. The second term in this equation does not contribute to the expression for $W_\alpha(\boldsymbol{x})$, since Δ vanishes on the spectrum.

5. Scars

The discussion so far was very general. It does not depend on what type of chaos, if any, exists in the underlying dynamical system. However, motivated by the existence of eigenstates, *scars*, which apparently are peaked on periodic orbits, we want to express the results as a sum over periodic orbits. This has so far only been done for the hard chaos case, although the results can no doubt be extended to include non-isolated orbits and integrable systems. Mixed chaos systems remain a serious challenge. In particular, it will be shown that (57) is equivalent to the result obtained by AF [12, 13].

Semiclassically, the dominant contribution is from regions where $\boldsymbol{r}'$ is small, so the orbits appearing begin and end near the phase space point $\boldsymbol{x}$. On the assumption that such approximately closed orbits are well represented by small deviations from *periodic* orbits passing near $\boldsymbol{x}$, it has been shown by Berry [10] that

$$W(\boldsymbol{x}, E) = W_0(\boldsymbol{x}, E) + \sum_j W_j(\boldsymbol{x}, E) \tag{64}$$

where W_j is the contribution of a periodic orbit and W_0 is the Weyl term coming from short direct trajectories. As discussed by [21], this approximation is likely to be good for hard chaos, but not so good for mixed chaos.

Berry's formula is not completely satisfactory since it does not converge for physical E. It also contains explicit terms which represent repetitions of primitive periodic orbits. Voros [5] demonstrated how explicitly summing out these repetitions helped to deal with the divergence difficulty. The contributions coming from periodic orbits will be described in what follows.

In particular, the contribution of the jth orbit in (64), that consists of a_p repetitions of the primitive orbit p, is [10]

$$W_j(\boldsymbol{x}, E) = W_{a_p}(\boldsymbol{x}, E) = -4\pi \mathrm{i} A(\boldsymbol{x}, E) \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_j(E)+Y_j)}}{\sqrt{\det(M_j+1)}} \,. \tag{65}$$

The monodromy matrix of the orbit $M_j = \left(M_p\right)^{a_p}$ has the eigenvalues, in two dimensions, $\mathrm{e}^{\pm a_p u_p}$ where u_p is the Lyapunov exponent of the primitive orbit, and a_p is the number of repetitions. For hard chaos all orbits are hyperbolic (without reflection, for simplicity). The action (that contains the Maslov index) is $S_j = a_q S_p$ and

$$Y_j = \tilde{\boldsymbol{X}} J \frac{M_j - 1}{M_j + 1} \boldsymbol{X} \,. \tag{66}$$

Here $\boldsymbol{X} = \binom{\delta p_\perp}{\delta r_\perp}$ is the vector distance from the phase space point $\boldsymbol{x}$ to the orbit p and $\tilde{\boldsymbol{X}}$ is its transpose. The symplectic matrix J is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The factor $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}Y}$ gives the fringes as the point $\boldsymbol{x}$ goes away from the orbit. It is assumed that orbits passing far from $\boldsymbol{x}$ will give such violent oscillations from this factor that they effectively can be neglected.

By assuming that the SS has been conveniently chosen, one can suppose that all periodic orbits cross the SS. The Weyl term W_0 is the contribution of direct trajectories in (37). The contribution from long orbits is

$$W'_r(\boldsymbol{x}, E) = \sum_j W_j(\boldsymbol{x}, E) \tag{67}$$

where the sum is now restricted to *periodic* orbits that cross the SS r times. Grouping the terms in this way guarantees convergence.

We wish to display the contribution of a given primitive orbit however. Introduce the notation to label composite orbits,

$$\mu = \left\{m_1, m_2, \ldots m_p, \ldots\right\} \tag{68}$$

where m_p is the number of repetitions of the pth primitive periodic orbit. All but a finite number of m's vanish. The composite length, namely the number of crossings of the surface of section, is

$$L_\mu = \sum_p m_p L_p \tag{69}$$

and the composite action is

$$S_\mu = \sum_p m_p S_p \,. \tag{70}$$

The composite quantities are thus expressed in terms of the primitive quantities. The orbital structure of W'_r is special, as just one orbit occurs in any given term W_j.

The terms in W'_r can be grouped now in a different way. We can write it as

$$W'_r = \sum_p \sum_{a_p} W_{a_p} \tag{71}$$

where p labels the primitive periodic orbits and a_p their repetitions. The constraint

$$a_p L_p = r \tag{72}$$

is imposed on the orbits in (71).

In order to explore the contribution of a specific primitive periodic orbit to the Wigner function it is instructive to group the terms in (50), (51) according to primitive orbits. The expansion coefficients of the Fredholm determinant D_n contain only composite orbits of length n. We introduce the notation $\mu(n)$ to label the general composite orbit of length n, i.e.

$$L_{\mu(n)} = n\,. \tag{73}$$

The form of the terms entering D_n of (11) implies that

$$D_n = \sum_{\mu(n)} C_{\mu(n)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\mu(n)}}\,. \tag{74}$$

The coefficients $C_{\mu(n)}$ can be explicitly found with the help of the recursion relation equation (14) and the well known explicit expression for the traces σ_r in terms of periodic orbits in the case of hard chaos. The resulting form is, $C_\mu = \prod_p C_{p,m_p}$, where $C_{p,0} = 1$ and for $m_p > 0$,

$$C_{p,m_p} = \sum_{\{m_p\}} (-1)^{\nu_p} \prod_{i=0}^{\nu_p - 1} \frac{1}{m_{p,i}} \frac{1}{\sqrt{|\det(M_p^{m_{p,i}-m_{p,i+1}} - 1)|}}\,. \tag{75}$$

The sum in (75) is over partitions of the integer m_p, where a partition is a set of integers $m_{p,i}$, $i = 0..\nu_p$, such that $m_{p,0} = m_p, m_{p,\nu_p} = 0$, and $m_{p,i} > m_{p,i+1}$. Note that C_{p,m_p} satisfies the recursion relation

$$C_{p,m_p} = -\frac{1}{m_p} \sum_{r=1}^{m_p} \frac{1}{\sqrt{|\det(M_p^r - 1)|}} C_{p,m_p - r}\,. \tag{76}$$

With the help of this decomposition (51) takes the form

$$U_n = \sum_{r=0}^{n} \sum_{\mu(n-r)} C_{\mu(n-r)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\mu(n-r)}} \sum_p \sum_{a_p} W_{a_p} \tag{77}$$

with the restriction that

$$L_p a_p = r + 1\,. \tag{78}$$

Because of equation (78) the sum over r can be replaced by a sum over a_p. Then equation (77) can be rearranged to take the form

$$U_n = \sum_p U_n^p \tag{79}$$

with

$$U_n^p = \sum_{a_p} \sum_{\mu(n-r)} C_{\mu(n-r)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\mu(n-r)}} W_{a_p}\,. \tag{80}$$

Since all the sums in question are finite there is no problem with the interchange of the order of the various summations. The equations (50), (79) and (80) form a 'scar' formula, namely an expression for the Wigner function as a sum over primitive periodic orbits. This is the main result of this paper.

In the appendix we manipulate (80) to show that it agrees with the result of AF. Substitution of (A20), (79) and (63) in (55) yields,

$$W(\boldsymbol{x};E)=\frac{\pi}{\Delta}\left[A(\boldsymbol{x},E)\Delta_i-B(\boldsymbol{x},E)\Delta\right] \\ -\frac{8\pi A(\boldsymbol{x},E)}{\Delta}\mathrm{Re}\left[\mathrm{i}\sum_p\sum_{\mu(n+1\leqslant N_1+1)}\sum_{k=0}^{\infty}c_\mu^{(p,k)}\mathrm{e}^{-\mathrm{i}\pi\mathcal{N}}g^{(k)}(b_p)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\mu+S_p)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X}}\right] \tag{81}$$

where $\mathcal{N}=\Phi/2\pi$ is the integrated smoothed density of states, while $\boldsymbol{R_p}$, b_p and g are defined in the appendix. Since the sum (55) is restricted to $n\leqslant N_1$ the sum over μ is restricted to orbits crossing the SS N_1+1 times. The formula for an eigenstate corresponding to (57) is

$$W_\alpha(\boldsymbol{x})=\frac{\pi A(\boldsymbol{x},E_\alpha)}{h^2\Delta'(E_\alpha)}\left\{\Delta_i-8\,\mathrm{Re}\left[\mathrm{i}\sum_p\sum_{\mu(n+1\leqslant N_1+1)}\sum_{k=0}^{\infty}c_\mu^{(p,k)}\mathrm{e}^{-\mathrm{i}\pi\mathcal{N}}g^{(k)}(b_p)\right.\right. \\ \left.\left.\times\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\mu+S_p)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X}}\right]\right\}_{E=E\alpha}. \tag{82}$$

This result is similar to the one found by AF (corrigendum in [12]). The difference is that the cut-off here is sharp while the cut-off by the error function Erfc used by AF is smooth. (Note for comparison that $\mathrm{Erfc}(-\infty)=2$.)

6. Conclusions

In this paper we have obtained formulae for the Wigner functions of the eigenstates of a closed system, as well as for the Wigner transform of the resolvent operator. These formulae are at several levels of generality and approximation. Formula (50) can be regarded as exact, if an exact version of the transfer operator $\boldsymbol{T}$ is employed. In principle it could be implemented numerically, although it is probably not the recommended procedure for exact numerical calculation.

If $\boldsymbol{T}$ makes quasiclassical errors, this formula in principle provides a way to calculate the Wigner functions for a general closed system. Once the quasiclassical approximation is made, it can be further exploited by using the fact that $\boldsymbol{T}$ is quasiclassically of finite rank and is unitary, except for quasiclassical errors, on the space where it does not vanish. This gives the most general, compact formula, equation (52) for the Wigner functions in the quasiclassical approximation.

The short orbit (Weyl) contributions can be worked out in detail in the general case. Also the contribution of the long orbits which by resurgence are like the short orbits can be carried out. This allows a general formulation of the contribution of the Weyl term, at least if the energy levels are independently known. It is found [22] that this contribution dominates the wavefunctions, with the contributions of longer orbits making relatively small corrections. In general, the spectral determinant Δ and its components $\tilde{\Delta}$ can be found, if necessary, by diagonalization of the finite dimensional T operator. Therefore, most of the weight of the wavefunction for arbitrary closed systems can be found by use of equation (54).

Practical quasiclassical calculations for the spectrum which are based completely on orbits have so far only been possible for certain special systems. These include hard chaos systems, in which all phase space points are hyperbolically unstable. Certain corrections and extensions can be made. For example, some non-isolated orbits can be included. Diffraction [23], refraction and ray splitting [24] effects can also be accounted for. Integrable systems

can also be handled by related methods [25]. All these methods are based on the periodic orbits of the system. They can be regarded as applications of the Fredholm method, used in the paper, to the calculation of the spectral determinant Δ and the Fredholm coefficients D_n in various special cases.

These results can be used in our formulae. However, part of the formula, namely $W'_r(\boldsymbol{x}, E)$, is expressed in terms of 'midpoint rule' orbits, which depend on the phase space point $\boldsymbol{x}$ at which the Wigner function is evaluated. Since finding the periodic orbits tends to be the most difficult part of the calculation, it is likely currently impractical to implement the even more difficult problem of finding 'midpoint rule' orbits in the general case.

However, for hard chaos, it is known how to express midpoint rule orbits in terms of nearby periodic orbits. Using these results, we obtain the formulae equations (79), (80) which are in a form that could be numerically implemented, given a knowledge of the periodic orbits. It is also a 'scar' formula since it is expressed as a sum of contributions each of which is associated with a given primitive periodic orbit. The contribution of such a term is large only near the orbit. Thus, it is possible to study how the eigenstates are 'scarred' by these orbits, that is, how they may be concentrated to some extent in the neighbourhood of such an orbit.

We have not used this formula for numerical calculations however, since an essentially equivalent formula has already been obtained and evaluated by AF. Equation (82) is (almost) the formula of AF. It was originally obtained by direct generalization of the techniques of Voros [5] and Berry and Keating [8] to the calculation of the Wigner functions. That method involves summing over repetitions of periodic orbits and showing that the result is quasiclassically the logarithmic derivative of a 'zeta function' expressed as an infinite product. Many special tricks were invented by AF to carry through this calculation for the Wigner function.

For completeness, we have shown that formula (50) of this paper is equivalent to that of AF (see (81) and (82)). The difference is that AF used Berry–Keating's version of resurgence which leads to a smooth cut-off of the contributions rather than a sharp cut-off at length N. We could have obtained that version by our method, as well, since we can show that the appropriate objects in (50) have the analyticity and functional requirements of Berry–Keating. However, this method does not explicitly show how the orbits of length N conspire to combine into contributions of orbits of length 0. For this pedagogical reason, we have chosen this version of resurgence in this paper.

Acknowledgments

This work was supported by National Science Foundation Grant DMR 9114328, and by the Israel–US Binational Foundation Grant (BSF), Jerusalem. REP expresses his thanks to Yossi Avron for hospitality at the Institute for Theoretical Physics of the Technion, Israel Institute of Technology, where this work was completed. We wish to thank Oded Agam for comments.

Appendix. Formula for U_n^p in terms of primitive periodic orbits

In this appendix an expression for U_n^p of (80) is derived. For this purpose the sum over repetitions a_p is explicitly performed. The derivation relies on the one of AF [12] and, in particular, on appendix C there. The derivation starts with the expansion of (65) in the form

(AF, equation (3.18)):

$$W_{a_p} = -4\pi \mathrm{i} A(\boldsymbol{x}, E) \sum_{k=0}^{\infty} g^{(k)}(b_p) t_p^{a_p} x_p^{k a_p} \mathrm{e}^{b_p} \tag{A1}$$

where (AF, equation (3.19))

$$b_p = b_p(\boldsymbol{x}) = \frac{\mathrm{i}}{\hbar} \tilde{\boldsymbol{X}} \boldsymbol{R}_p \boldsymbol{X} \tag{A2}$$

and (AF, equation (3.20))

$$g^{(k)}(b_p) = \sum_{l=0}^{k} (-1)^{k-l} f_p^{(l)} \tag{A3}$$

while (AF, equation (3.17))

$$f_p^{(l)} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} \tilde{\boldsymbol{X}} \boldsymbol{R}_p \boldsymbol{X}} \frac{1}{l!} \left(\frac{\partial}{\partial z} \right)^l \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \frac{1-z}{1+z} \tilde{\boldsymbol{X}} \boldsymbol{R}_p \boldsymbol{X}} \Bigg|_{z=0} . \tag{A4}$$

The matrix $\boldsymbol{R}_p$ is defined by the equation (AF, equation (3.13))

$$Y_j = \tilde{\boldsymbol{X}} \boldsymbol{J} \frac{\boldsymbol{M}_p^{a_p} - \boldsymbol{I}}{\boldsymbol{M}_p^{a_p} + \boldsymbol{I}} \boldsymbol{X} = \tanh\left(\frac{u_p a_p}{2} \right) \tilde{\boldsymbol{X}} \boldsymbol{R}_p(\boldsymbol{x}) \boldsymbol{X} \tag{A5}$$

where $t_p = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_p - \frac{1}{2} u_p}$ and $x_p = \mathrm{e}^{-u_p}$, while the Maslov index was absorbed in the action. Both $\boldsymbol{R}_p$ and b_p depend only on the primitive orbit and not on its repetitions.

The above, rather complicated, formulation was found necessary to be able to sum over the orbit repetitions, a_p, contained in the factor $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} Y_j}$. The expansion of this factor is not necessary if the methods of the present paper are used.

In order to calculate the sum (80) we first expand (see AF, equations (2.7) and (C.3))

$$C_\mu \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_\mu} = \prod_p d_{m_p}(x_p)(-t_p)^{m_p} \tag{A6}$$

where the $\{m_p\}$ are related to μ by (68), while (AF, equation (C.2))

$$d_m(x) = \sum_{j_1 > j_2 > \cdots > j_m} x^{j_1 + j_2 + \cdots + j_m} = \frac{x^{m(m-3)/4}}{(x^{-1/2} - x^{1/2})(x^{-1} - x^1) \cdots (x^{-m/2} - x^{m/2})} \tag{A7}$$

with $d_0(x) = 1$. This result was based on the zeta function formula of Voros [5]

$$D(E) = \det(1 - \lambda \boldsymbol{T}) = \prod_p \prod_{m_p} (1 - \lambda^{L_p} t^p x_p^{m_p}) . \tag{A8}$$

The product over m_p is done by the Euler product formula.

The sum (80) takes the form

$$U_n^p = \sum_{a_p} \sum_{\mu(n-r)} \prod_q d_{m_q}(x_q)(-t_q)^{m_q} W_{a_p} \tag{A9}$$

where r is given by (78) and the $\{m_q\}$ are related to $\mu(n-r)$ by (68) and (73). The first term in the sum over a_p is $a_p = 1$, which corresponds to the integer $r_p = L_p - 1$. The corresponding composite orbit is

$$\mu(n - r_p) = \{\bar{n}_1, \bar{n}_2, \bar{n}_3, \ldots, \bar{n}_p - 1, \ldots\} . \tag{A10}$$

In general, the sum over composite orbits contains terms of the form

$$\mu(n - r) = \{\bar{n}_1, \bar{n}_2, \bar{n}_3, \ldots, \bar{n}_p - a_p, \ldots\} \tag{A11}$$

where

$$\mu(n+1) = \{\bar{n}_1, \bar{n}_2, \bar{n}_3, \ldots, \bar{n}_p, \ldots\}\,. \tag{A12}$$

Note that all the sums for U_n^p include only composite orbits where n_p varies and all the other $n_i = \bar{n}_i$ are fixed. With the help of (A1) and (A6) the sum (A9) takes the form

$$U_n^p = -4\pi \mathrm{i} A(\boldsymbol{x}, E) \sum_{\mu(n+1)} \sum_{k=0}^{\infty} \bar{C}_\mu^{(p,k)} g^{(k)}(b_p) \mathrm{e}^{b_p} \tag{A13}$$

where

$$\bar{C}_\mu^{(p,k)} = \prod_{q \neq p} [d_{\bar{n}_q}(x_q)(-t_q)^{\bar{n}_q}] \tilde{d}^{(p,k)} \tag{A14}$$

and

$$\tilde{d}^{(p,k)} = \sum_{a_p=1}^{\bar{n}_p} d_{\bar{n}_p - a_p}(x_p)(-t_p)^{\bar{n}_p - a_p} t_p^{a_p} x^{k a_p}\,. \tag{A15}$$

Following AF it is useful to introduce (AF, equation (C.4))

$$d_m^{(k)}(x) = \sum_{\substack{j_1 > j_2 > \cdots > j_m \\ j_1, j_2, \cdots j_m \neq k}} x^{j_1 + j_2 + \cdots + j_m} \tag{A16}$$

that satisfy (AF, equation (C.9))

$$d_m^{(k)}(x) = \sum_{j=0}^{m} (-1)^j x^{jk} d_{m-j}(x)\,. \tag{A17}$$

With the help of this equation one finds that

$$\tilde{d}^{(p,k)} = t_p(-t_p)^{\bar{n}'_p} x_p^k d_{\bar{n}'_p}^{(k)}(x_p) \tag{A18}$$

where $\bar{n}'_p = \bar{n}_p - 1$. With the help of (C.11) of AF one finds

$$\bar{C}_\mu^{(p,k)} = \prod_{q \neq p} d_{n_q}(x_q)(-t_q)^{\bar{n}_q}(-t_q)^{\bar{n}'_p} t_p x_p^k d_{\bar{n}'_p}^{(k)}(x_p) = c_{\mu(n+1)}^{(p,k)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\mu + S_p)} \tag{A19}$$

where $c_\mu^{(p,k)}$ are defined in AF, equations (C.11) and (3.29). Substitution in (A13) yields the final result

$$U_n^p = -4\pi \mathrm{i} A(\boldsymbol{x}, E) \sum_{\mu(n+1)} \sum_{k=0}^{\infty} c_\mu^{(p,k)} g^{(k)}(b_p) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\mu + S_p)} \mathrm{e}^{b_p}\,. \tag{A20}$$

References

[1] Einstein A 1917 *Verh. Dtsch Phys. Ges.* **19** 82

[2] Gutzwiller M C 1967 *J. Math. Phys.* **8** 1979; 1969 *J. Math. Phys.* **10** 1004; 1970 *J. Math. Phys.* **11** 1791; 1971 *J. Math. Phys.* **12** 343

[3] Balian R and Bloch C 1970 *Ann. Phys., NY* **60** 401; 1971 *Ann. Phys., NY* **63** 592; **64** 271; 1972 *Ann. Phys., NY* **69** 76; 1974 *Ann. Phys., NY* **85** 514

[4] Gutzwiller M C 1991 *Chaos in Classical and Quantum Mechanics,* (New York: Springer); 1991 *Chaos and Quantum Physics (Proc. Les Houches Summer School, Session LII*) ed M J Giannoni, A Voros and J Zinn-Justin (New York: Elsevier)
Eckhardt B 1988 *Phys. Rep.* **163** 205

[5] Voros A 1988 *J. Phys. A: Math. Gen.* **21** 685

[6] Doron E and Smilansky U 1992 *Phys. Rev. Lett.* **68** 1255; 1992 *Nonlinearity* **5** 1055
Dietz B and Smilansky U 1993 *Chaos* **3** 581

[7] Cvitanović P and Eckhardt B 1989 *Phys. Rev. Lett.* **63** 823
[8] Berry M V and Keating J P 1990 *J. Physique* **23** 4839; 1992 *Proc. R. Soc.* A **437** 151
[9] Heller E J 1984 *Phys. Rev. Lett.* **53** 1515
Bogomolny E B 1988 *Physica* **31D** 169
[10] Berry M V 1989 *Proc. R. Soc.* A **423** 219; *Proc. R. Soc.* A **424** 279
[11] Stein J and Stöckman H-J 1992 *Phys. Rev. Lett.* **68** 2867
Sridhar S and Heller E J 1992 *Phys. Rev.* A **46** 1728
[12] Agam O and Fishman S 1993 *J. Phys. A: Math. Gen.* **26** 2113; corrigendum 1993 *J. Phys. A: Math. Gen.* **26** 6595
[13] Agam O and Fishman S 1994 *Phys. Rev. Lett.* **73** 806
[14] Georgeot B and Prange R E 1995 *Phys. Rev. Lett.* **74** 2851
[15] Georgeot B and Prange R E 1995 *Phys. Rev. Lett.* **74** 4110
[16] Bogomolny E B 1992 *Nonlinearity* **5** 805
[17] Smithies F 1962 *Cambridge Tracts in Mathematics and Mathematical Physics* vol 49 (Cambridge: Cambridge University Press)
[18] Van Vleck J H 1928 *Proc. Nat. Acad. Sci., USA* **14** 178
[19] Prozen T 1994 Exact quantum surface of section method *Preprint*
[20] Georgeot B and Prange R E *Fredholm Theory and the Quasiclassical Approximation* to be published
[21] Guo Y 1993 *PhD Thesis*
[22] Agam O Private communication.
[23] Vattay G, Wirzba A and Rosenqvist P R 1994 *Phys. Rev. Lett.* **73** 2304
Pavloff N and Schmit C 1995 *Phys. Rev. Lett.* **75** 61
[24] Prange R E, Ott E, Antonsen T M, Georgeot B, and Blümel R 1995 Smoothed density of states for problems with ray splitting *Preprint*
[25] Berry M V and Tabor M 1977 *J. Phys. A: Math. Gen.* **10** 371